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N. E. JOUKOVSKY

(1847 — 1921)

In this great personality we find the union of higher mathematics and engineering. In him we have the finest blend of theoretical science and technics; he was almost a university in himself.

(S. A. Chaplygin)

Nicholai Egorovich Joukovsky was born in 1847 in the little town of Orekhovo, near Vladimir. His father studied at the St. Petersburg Institute of Transport Engineers at a time when such outstanding scientists as Clapeyron and Lamé were on the teaching staff. When the time came to choose a teacher for his son, the father engaged a young Moscow physician, named Repman. His father's example, Repman's instruction, the atmosphere of learning in his home, all combined to mould the boy's character and develop him along scientific lines.

At the Moscow University, which he attended from 1864 to 1868, Joukovsky looked upon his studies as preparation for an engineering career. This interest in the practical aspects of his work must be regarded as an important factor in the development of his scientific thought.

The training he received in the classical university tradition of the 60's and 70's, when interest in mechanics centred about mathematical questions connected with the motion of solids influenced Joukovsky's first studies. However, even in these classical investigations, Joukovsky's work is already characterized by geometrical and physical content.

At the beginning of the 20th century we find the applied element in research fully developed in Joukovsky's studies in mechanics and hydrodynamics, lending his investigations a concreteness whose significance cannot be overestimated. Even as a scientist of world fame, Joukovsky spoke of the importance of engineering. Time and again we find the stimulus for abstract thought arising out of direct observation of natural phenomena.

Joukovsky's method introduced a new element in investigations, by making working hypotheses as a preliminary to the study. This enabled him to concentrate on the central feature of the problem, to simplify the study of complex phenomena. However, once the problem was outlined, it was pursued with all the painstaking and rigorous mathematical analysis required by Newtonic thought.

Joukovsky drew his working hypotheses from experiment; experiment again was the final test of all theoretical results. In this sense he was the forerunner of the Goettingen school; essentially the same idea in mechanics research is now followed all over the world.

After presenting his master's thesis in 1876, Joukovsky made his first trip abroad. It was in the course of this sojourn, spent mainly in France, that he made the acquaintance of the brilliant French school of the day: of Darboux and Poincaré, of Lévy and Résal. How these contacts influenced him is seen in later classical studies in mechanics, particularly in the stability of motion.

In Paris Joukovsky came more deeply in touch with the geometry of Poncelet and Monge; his studies at the time strengthened the tendency toward geometrical exposition seen in his master's thesis *Kinematics of Fluids*; and made for the geometrical elegance characterising his exposition throughout his life.

Exactly when Joukovsky first became interested in flying is impossible to say. Perhaps when, as a boy, he stood watching the birds wheeling above the Orekhovo fields. We know he attended international aeronautical congresses; that during a vacation period from the University he entertained the country lads by explaining the features of the flight of the pigeons.

In 1891 he published his first study *The Soaring of Birds*. Others had dreamed of flying: the tale goes back at least as far as the Daedalus and Icarus myth. Joukovsky's was not a mere dream; it was amazing scientific intuition coupled with the knowledge of physics and mechanics required to give it concrete existence.

In 1895 Joukovsky heard of Lilienthal's glider and his experiments in flying in a heavier-than-air craft. His boundless curiosity and innovator's enthusiasm were roused to the highest pitch: he traveled to Lilienthal's home, made his acquaintance, and spent some time studying the glider and discussing its construction and possibilities. By 1902 he had constructed a wind tunnel in the mechanics laboratory of the Moscow University and was conducting experiments on the lift force and resistance of a wing. The significance of these first studies and experiments is all the greater when we recall that the first successful flight of the Wright brothers was not carried out until 1903.

In 1906 Joukovsky published his investigation *Bound Vortices*, containing his famous theorem on the connection between the lift force and circulation. In subsequent studies of the origin of circulation, Joukovsky evolved the theory that it is the streams breaking away from the back edge of the wing that give rise to the circulation. The Joukovsky hypothesis made it possible to determine the circulation about a wing and gave the Joukovsky theorem its direct application to aviation. The successful studies of lift force led to both Joukovsky's theory of the propeller and Prandtl's theory of the finite wing: both theories as well as the determination of lift force are the development of the famous Joukovsky theorem. Joukovsky's work in the field laid the foundations of aerodynamics as a science.

An examination of Joukovsky's achievements in aviation increases our astonishment that he should have done so much in other fields. He carried out geometrical analyses in the impact of solids and presented new conceptions of degrees of stability of motion. His studies of solids containing cavities filled with a liquid are of the highest interest to astronomy and laid the ground for further developments in the theory of modern shells. He studied the rolling of ships and properties of lubricants. He investigated the nature of flow in rivers, the shape of snowdrifts and silt deposits. He explained the movement of ground water and the factors causing water hammer in pipes. Nor was this diffusion of interest, that so often leads to superficiality. It was the application of rare genius to various natural phenomena; and in practically every field he touched upon, Joukovsky's achievements are outstanding.

Joukovsky was truly indefatigable in research. He returned to the same problems time and again. He was never satisfied until he was in a position to present his results in final form, with numerical examples: we are struck by their geometrical elegance and by the striking analogies his great erudition enabled him to draw. His inexhaustible energy of mind produced as high as four profound studies a year; the ten volumes of his collected works contain numerous fundamental investigations and a great number of shorter articles and speeches, all of high excellence.

Joukovsky's work lives today and will continue to be of significance in the future. Every examination brings to light new wellsprings of thought and stimulates new lines of research. Each succeeding generation of engineers finds something it can take from Joukovsky, and this process is far from ended. He is not only the teacher of Chaplygin and numerous other outstanding scientists in this country, not only the pride of Russian science: the entire world is indebted to him for a great and lasting contribution to human progress.

THE WORKS OF JOUKOVSKY

Joukovsky's investigations in the field of mechanics are so varied and extensive that it is possible to outline only his basic researches here. It has been found convenient in the present review to group his works under three main headings: hydrodynamics and hydraulics, general mechanics, and aerodynamics; chronological order being preserved where possible[1].

1. Hydrodynamics and Hydraulics

Joukovsky's work in hydrodynamics began with his master's thesis *Kinematics of Fluids* [1, III] presented in 1876 at the Moscow University.

Investigation of deformation of fluid particles in motion had not been carried to completion at the time. Research was mainly of an analytic character and therefore lacked the clarity brought into any field by geometry.

Joukovsky himself states in the introduction to his thesis:

"The great clarity which geometrical investigation lends to the study of the dynamics of solids leads us to expect significant success in hyrodynamics through a study of the kinematics of variable systems".

Chapter I of the thesis gives the results obtained by Cauchy and Helmholtz, and goes on to a discussion of the motion of an infinitesimal fluid particle. The motion of the particle is resolved into external translation and rotation and internal strain motion, changing the originally spherical shape into an ellipsoid.

Joukovsky resolves the internal motion of the particle into elongation, shear and deviation; and goes on to a determination of lines and planes of constant direction for total (internal and external) motion of the particle relative to its centre. Joukovsky proves that the locus of normals to planes of constant direction in strain and rotation of the particle is a cone of the third order. The generatrix of the cone passes through the axes of deformation; the intersections of the cone and the planes of deformation are projections of the axis of rotation.

[1] The Arabic numerals in brackets following titles indicate works in the chronological list (page 27), while the Roman numerals indicate the volume of the Collected Works of Joukovsky where the given work may be found.

Chapter two discusses stream lines and the velocities of the fluid. The investigation of streamline surfaces leads to Dupin's theorem of orthogonal surfaces, and to Joachimstahl's theorem. In the subsequent discussion of circulation of velocities the author introduces critical points, in whose vicinity the shape of lines of flow is discussed in great detail. At these points the streamlines either intersect, osculate or have infinite curvature, the velocity components are either equal to zero or infinity, or become indeterminate.

In the investigation of plane parallel motion, Joukovsky makes a detailed analysis of the distribution of streamlines in the vicinity of critical points. It is found that in the vicinity of zero critical points of the first order, i. e. points at which velocity components are equal to zero while their derivatives relative to the coordinates are finite and not all equal to zero, the streamlines are either hyperbolic or elliptic. In the former case the two streamlines intersect in the critical point, in the latter case the streamlines pass around the critical point, approaching infinitesimal ellipses. Zero points of the $n-1$ order or infinite points of the $n-1$ order, should they exist, are intersection points of n streamlines.

The chapter ends with a discussion of certain properties of a three-dimensional non-vortex motion of an incompressible fluid, and of a related motion whose velocities are determined by the curvature of the streamlines and the contraction of the currents. Chapter three is devoted to a discussion of the resultant flow for given components, and the theorem of Beltrami.

Chapter four investigates properties of accelerations of points of a fluid in motion. The chapter opens with general theorems on acceleration, among them a generalization of the theorem of Coriolis. This is followed by a discussion of conditions for trajectories and velocities of flow of an incompressible fluid, for which the acceleration will have a potential function.

In his works *Reaction in Cases of Expulsion and Inflow of a Liquid* [13, 14, IV] (1882, 1885) Joukovsky proves that the reaction on a stationary vessel immersed in an infinite liquid is equal to zero in both cases, provided there is no friction and the variation of the velocity is continuous. Where the expulsion is accompanied by the formation of a jet, on whose surface variation of velocity is discontinuous, the reaction is finite. The value of this reaction, determined by the author, remains the same even if the vessel is not stationary. The author also shows that the reaction of inflow is not equal to zero in the case of a non-stationary vessel.

The paper, *Impact of Two Spherical Solids, One of Which is Floating in a Liquid* [22, III] was published in 1884. One ball, floating in the centre of a semi-spherical vessel receives an impact from a second falling vertically. Joukovsky takes up the problem for both elastic and inelastic spheres, and shows that the formulae for the velocities after impact are the same as for impact in a vacuum, provided the mass of the floating ball is increased to definite proportions. If the mass of the liquid is infinite, the mass of the floating ball must be increased to half the mass of the displaced liquid. This work was the first of a large number by Russian scientists devoted to the impact of solids on water.

An extensive work by Joukovsky, *Motion of a Solid with Cavities Filled with a Homogeneous Liquid*, [27, III] was published in 1885. The work was awarded the Brashman prize by the Moscow University in 1886. Existing works at the time either dealt with particular cases or only touched on the problem in passing.

Joukovsky's is the first general treatment of the subject. In addition to several new general theorems, it contains solutions of the problem for a number of cavity shapes not investigated previously, and for multi-connected cavities. Joukovsky extends his investigation to cases of a viscous fluid and to vortex motion.

In its most general form, motion is composed of translation and rotation. Translation does not give rise to motion of the fluid relative to the cavity; whereas rotation causes relative motion of the fluid enclosed in the cavity, this motion being completely determined by the rotation of the body. In case of the existence of a velocity potential, the determination of the motion of the liquid is analytically reduced to the internal problem of Neumann for the Laplace equation. To obtain the equation of motion of the solid, the solid and the liquid may be regarded as a single dynamic system, when the motion of the liquid has been determined in terms of velocity components.

In chapter one the author shows that in the relative motion of the liquid, velocity is greatest at the surface of the cavity, and that if velocity of relative motion at any point in the liquid becomes very great, the pressure throughout the volume also becomes very high. Joukovsky goes on to show that without changing the motion of the system as a whole, the liquid mass may be substituted by an equivalent solid[1]; should the solid be one with multi-connected cavities and if the liquid is given an initial circulation in the cavities, in addition to replacing these liquid masses by equivalent solids a gyroscope must be attached, the direction of whose axis of rotation and moment of initial momentum are determined by the principal moment of momentum of the liquid masses when the body is at rest. Joukovsky points out that his method of treating cases of multi-connected cavities removes the doubt felt by Neumann in investigating the problem.

Chapter two deals with the determination of the internal motion of the liquid and of the ellipsoids of inertia of equivalent solids for cavities of various shapes. It opens with a detailed treatment of an ellipsoidal cavity, then goes on to cylindrical and prismatic cavities. Joukovsky then proceeds to a discussion of cavities having the shape of solids of revolution, the entire system itself rotating about an axis at right angles to the axis of the cavity. As an example of a multi-connected system, the author considers a bi-connected cavity as represented by an infinitely thin closed tube of arbitrary shape. He shows that for all thoroidal cavities the principal moment of the intial momentum is equal to the product of **the** mass of the liquid and the circulation velocity, divided by 2π.

The first case considered in chapter three is that of vortex motion of the liquid in the cavities of a moving body. Joukovsky sets up equations which must be satisfied by the components of the angular velocity of the solid and by the components of the vortex. He also establishes the formula for the pressure of the liquid. As examples, he considers a cylindrical and an elliptic cavity.

Joukovsky now proceeds to a consideration of the influence of friction, setting up differential equations for the motion of solid and liquid, and the formula for the pressure. Joukovsky discusses Helmholtz' problem of oscillations of a solid about a fixed axis. The solid contains a viscous fluid within a spherical cavity and is subjected to a torque proportional to the angle of rotation.

[1] Stokes' proof of the possibility of replacing a liquid mass by an equivalent solid refers only to the particular case of low velocities. Joukovsky's is the general case.

Employing Poiseuille's theory of laminar flows, the author solves a new problem, that of the motion of a closed tube filled with a liquid. The solution is verified by means of an apparatus constructed by the author himself.

The paper concludes with a number of remarks regarding the motion of a free solid containing a viscous fluid confined within it. Joukovsky proves that if initial velocities are imparted both to a solid containing a cavity and to the viscous liquid filling the cavity, the motion of the system tends to an ultimate state in which one of the principal axes of inertia has the direction of the principal moment of the initial moments of motion, and the whole system will rotate about this axis with a constant angular velocity. This result is of great importance in cosmogony.

We are indebted to N. Joukovsky for important results in the hyrodynamic theory of friction. The first of three works he wrote on the subject is *Hydrodynamic Theory of Friction of Well Lubricated Solids* (1886). Joukovsky's interest in the subject was aroused by a previous work by N. P. Petrov, who investigated the motion of the lubricant layer between rotating concentric cylinders. Petrov held that the hydrodynamic pressure over this layer is constant. However, Joukovsky points out the necessity of considering the forces enabling the lubricant to counteract the load exerted upon it by the shaft. Joukovsky considers these forces as pressure resulting from the motion of a thin layer of lubricant confined between rotating eccentric cylindrical surfaces. In view of the complexity of the problem, Joukovsky in this work limits his investigations to a somewhat different case. In the conclusion he points out that in general the action of the lubricant must be considered hydrodynamically, the more so since experiment proves the dependence of the thickness of the layer on the velocity. His second paper on the subject is *Motion of a Viscous Fluid Confined Between Rotating Eccentric Cylindrical Surfaces*[32, IV] (1887). Employing circular bi-polar Neumann coordinates and neglecting the force of inertia in comparison with the force of friction, Joukovsky reduces the problem to the solution of an equation in partial differentials of the second order. Hefinds both the velocity components within the layer, the pressure of the layer on the internal cylinder, and the resultant torque. The results are applied to the investigation of the rotation of a shaft in a bearing, when shaft and bearing rotate in opposite directions with equal angular velocities.

The investigation reached its final form in a third work which Joukovsky wrote in collaboration with his pupil S. Chaplygin, entitled *Friction of a Lubricant Between a Shaft and its Bearing*[101, IV], published in 1906. The method employed in this work is, in effect, the same as that in the previous paper. The authors succeed in obtaining a complete solution of the two-dimensional problem of the motion of a viscous fluid layer between eccentric circumferences, no particular assumptions being made as to the thickness of the layer. The authors show at the end of the work that the results of Sommerfeld (1904) for a layer of small thickness follow as a particular case. It is interesting to note that in 1924 Düffing, in Zeitschrifft für angewandte Mathematik und Mechanik, repeated this investigation.

In his work *Theoretical Investigation of the Motion of Ground Water*[36, VIII] (1889), Joukovsky takes up various cases of motion of water in sand. Employing the Darcy law, he sets up equations for this motion, neglecting inertia terms. The author shows that the motion of water in sand is governed by the same laws which hold for the propagation of heat, the piezometric head and discharge acting as the temperature and quantity of heat respectively. The paper takes up various cases of wells and reservoirs.

The object in this work is to find to what extent the theory based on the Darcy law agrees with practical results; since a number of investigators had found it necessary to make corrections in the law. Joukovsky shows that the Darcy law is fully satisfactory provided all the boundary conditions are taken into consideration, and if the law is not applied in cases of great distances from the well.

Connected with this paper is Joukovsky's experiment *Influence of Pressure on Sand Saturated with Water*[40,VII] (1890) in which he shows that the level of the water in sand rises with an increase in atmospheric pressure. The phenomenon is due to the presence in the sand of air bubbles whose volume diminishes as pressure increases. This enables the water to occupy more space.

Joukovsky outlines the general method for the investigation of filtration of a liquid in his last work, published after his death, *Filtration of Water through a Dam*[157,VII] (1923). He gives a concise description of how the procedure given in his *Modification the Kirchhoff Method...*, may be applied to arrive at the solution of numerous problems in filtration earlier investigated by N. N. Pavlovsky.

In a concise paper *On the Form of Ships' Hulls*[41,IV], 1890 Joukovsky applies the Rankin frictional theory of force of resistance to the shape of the hulls of ships with a sharp bow. Joukovsky considers the problem as a case of plane parallel flow and introduces elliptical coordinates. The author expresses the rectilinear coordinates of points on the contour of the ship in terms of these elliptical coordinates by means of infinite series. He selects the coefficients of these series to arrive at a suitable contour which is simultaneously the stream line.

Joukovsky's interest in the calculation of water waves and the rolling of a ship led him in 1908 to make reports on the question to the Naval Department. Unfortunately, the text of these reports has been lost. However, A. P. Kotelnikov came upon a draft manuscript among Joukovsky's papers, entitled *On a Body Floating on the Surface*[162,IV]. While this work published by A. P. Kotelnikov in 1931 may not be the final variant, it is of great interest as the first attempt at a hydrodynamic solution of the problem, pointing out for the first time the importance of considering the conjoint mass of liquid in calculating the rolling of ship.

In another work entitled *The Wave in a Ship's Wake*[114,IV](1907) Joukovsky studies the shape of the wave in the wake of a ship of cylindrical form. He determines the resistance due to the wave, analysing the phenomenon both in shallow and in deep water. Neglecting the vertical component of acceleration and assuming that the distribution of pressure along the vertical is hydrostatic, Joukovsky reduces the problem to a differential equation of the second order, in partial derivatives, similar to the equation for oscillating motion. He obtains a formula for the resistance of a ship in the form of an integral including a function determining the shape of the ship's hull. Employing variation calculus, Joukovsky is thereby enabled to determine that form which will present the least resistance to motion. The form is found to consist of two arcs of a parabola.

We now come to a very important work, entitled *Modification of Kirchhoff's Method of Calculating a Two-Dimensional Motion of a Fluid, Given a Constant Velocity along an Unknown Streamline*[37,III](1890). Joukovsky himself formulates his objective as follows in opening paragraph:

"The proposed modification of Kirchhoff's method proceeds to the solution of a particular problem without preliminary conformal representation of the domains of the corresponding contours, thus eliminating a superfluous operation necessary in Kirchhoff's original method. The problem is restricted neither by the number of points of zero velocity nor by the number of streams".

Joukovsky's method considerably expands the field of the problems of plane parallel non-vortex flow of an incompressible fluid in cases of formation of jets. The method, which is now generally known, may be outlined in brief here. The author considers the function of the complex variable, the real part of the function being the velocity potential and the imaginary part, the streamline function. Joukovsky also considers the logarithm of the derivative of this function. The introduction of the logarithm presents a considerable advantage over the derivative alone, as found in Kirchhoff's method. The real part of the logarithm is equal to the logarithm of the ratio of the velocity at infinity to the velocity at the given point, while the coefficient of i is equal to the angle between the velocity vector and the real axis. Joukovsky maps the domains of these two functions of the complex variable on the upper semi-plane of an auxiliary complex variable so that the boundary streamline corresponds to the real axis. Expressing both functions of the complex variable in terms of the auxiliary variable, separating the real and imaginary parts and equating them to arbitrary constants, Joukovsky obtains two families of mutually orthogonal isothermic lines, which he calls the generating and the directing network respectively.

On the basis of these ideas, Joukovsky solves a large number of problems, the majority of which had never been previously investigated. The first group deals with the expulsion of a fluid from vessels, both symmetrical and asymmetrical; from vessels with infinite flat walls and from a vessel of finite width and infinite height or vice versa. In this group, though differing in some aspects, is the problem of expulsion of a fluid from an infinite vessel through a tube.

The second group takes up the impact of a jet on a symmetrical and on an asymmetrical wedge; symmetrical impact on a plate; impact on a plate consisting of two planes placed at an angle to each other, the initial flow being parallel to one of the planes; impact of a stream bounded by parallel walls against a symmetrical wedge; impact on a rectangular vessel; impact on a plate at the mouth of a channel.

The third group deal with the impact of a jet on a symmetrical wedge; the mutual impact of two jets; impact on a plate at the outlet from a channel with parallel walls; concentration of a jet by a funnel placed in its path.

The work ends with a discussion of a stream passing through a grid. Chaplygin and others of Joukovsky's pupils found it necessary to modify the results, which Joukovsky believed might apply to the action of a turbine. Chaplygin pointed out that the oncoming stream has a direction at right angles to the grid at infinity.

Joukovsky's method has been extended to cover a multitude of problems. It has also been generalized by *L. Sedov* in *Theory of Plane-Parallel Motions of a Perfect Non-Compressible Fluid* (1939) for flow past an arbitrary curvilinear contour and makes it possible to solve problems with any number of jets.

Joukovsky devoted a number of investigations to the theory of vortices, among them such important works as: *The Problem of Cutting Vortex Filaments*[58,III](1893), *Bound Vortices*[106,5](1907), *Karman's Vortex Theory of Drag*[133,V] (1914).

In the first of the above works, Joukovsky discusses the impossibility, established by experiment, of cutting a vortex filament into parts. In his theoretical investigation of the question, Joukovsky employs the method of conformal representations to study the motion of a vortex in the vicinity of a straight boundary of infinite length in the two following cases: when the fluid is at rest at infinity; and when the fluid moves along the boundary with a velocity at infinity.

The first case is investigated by connecting the representation to the vortex, the circulation being in the opposite direction; while the second case is derived from the first by superimposing an additional translation velocity on the entire mass of the fluid parallel to the real axis, two zero critical points being formed on this axis. By mapping the semi-plane on an angle, the author arrives in the first case at a motion of the vortex within the angle which is less than π, and in the second case a motion outside the acute angle. In the particular case, this angle may approach zero, and thereby represent the form of a knife edge. It is found that all the possible trajectories of the vortex deviate from the edge, proving that it is impossible to cut a vortex filament lengthwise.

Of the highest significance is the second work *Bound Vortices* (1907). We shall not give a detailed description, since the basic results are found today in all courses in hydrodynamics and aerohydrodynamics, including the well known Joukovsky theorem, which will be taken up in the section on aerohydrodynamics.

Published for the first time in Russia in 1890 and subsequently in French and English scientific journals is Joukovsky's important research, *Water-Hammer in Pipes*,[84, VIII] which is of great importance both to theory and practice.

Joukovsky founded his theoretical investigations on experiments carried out on the water system of the city of Moscow. Before Joukovsky, the formation of a water hammer by the rapid closing of a tap was not related to the theoretical investigation of the propagation of a sound wave in an elastic tube. Joukovsky points out that the probable reason why engineers had not thought of studying the problem in connection with wave propagation was that observations had been carried out only in short pipes, in which the rapidity of propagation of the wave made the phenomenon almost simultaneous throughout the range of the pipe. His own conception of the phenomenon enabled him to connect his research with the theoretical investigations of Résal, Gromecka, Korteweg and Lamb, on the variation of pressure along a pipe with elastic walls.

After formulating the general theory of the water hammer, Joukovsky applies it to such subjects as water hammer in a blind pipe, the reflection of the shock wave by the open end of a pipe, the influence of the time element in closing the tap on the water hammer, the effect of air cowl and water cowl, of safety valves. At the end of his work, Joukovsky gives the application of water hammer diagrams in determining the place of formation of air cushions or leaks in the pipe.

Joukovsky summarized his principal results as follows: 1. The water hammer is propagated along the pipe with a constant velocity, whose magnitude is not dependent to any significant degree on the force of the shock wave. This velocity is dependent on the material of which the pipe is constructed and the ratio of the thickness of its walls to its diameter. The velocity of the shock wave remains the same whether it is caused by the sudden stoppage of the flow of water in the pipe, or by a sudden great increase in the pressure.

2. The water hammer is propagated along the pipe with constant force. Its magnitude is proportional to the velocity of flow lost in the shock and the velocity of propagation of the shock wave in the pipe.

3. The rise of periodical oscillations of shock wave pressure is the result of the reflection of the shock wave by the ends of the pipe.

4. A dangerous increase in shock wave pressure occurs when the wave passes from a pipe of large diameter into a pipe of small diameter. When the wave reaches blind ends its force is doubled. This doubling may be repeated several times over, attaining great proportions should the corresponding conditions exist.

5. The simplest means of guarding against water hammer is the use of appliances to slow down the shutting off of the flow. The time spent in closing the tap should be proportional to the length of the pipe. Air cowls of the proper dimensions at the taps reduce the shock wave to negligible proportions if placed in the line of the pipe; however it is extremely difficult to preserve the air cushion in the cowl.

Of the great number of other works by Joukovsky on the subject of hydrodynamics and hydraulics, the two following will illustrate the breadth of his interests in the field; *Snow Drifts*[122,III] 1911, and *Snow Drifts and Depositing of Silt in Rivers*[150,III] 1919.

In snowstorms, drifts formed when the wind encounters an obstacle do not lie up against the obstacle, but leave a depression between the drift and the windbreaker. Joukovsky gives a general explanation of the phenomenon. In the first of the above works he considers plane parallel flow of a fluid, the obstacle being taken in the form of an infinite circular cylinder at right angles to the flow. Assuming the velocity of the wind to be proportional to the height above the earth, Joukovsky superimposes two flows on the given motion, chosen to make the contour of the cross section of the cylinder the stream line. These two flows have poles of the first and second order within the circular contour. Superimposing on the resultant motion another flow caused by a vortex to windward of the cylinder and by the representation of the vortex relative to the circular contour, Joukovsky arrives at streamlines corresponding to the motion of snow particles in the storm.

In the second paper, Joukovsky employs the expression for the streamline function in the vicinity of a zero critical point (given in his master's thesis) to set up the differential equations of motion of a snow particle caused by the force of gravity and the force of resistance of the air, the force of air resistance being taken proportional to the velocity of the wind related to the flying particle. By integrating these differential equations, Joukovsky defines the trajectory of the snow particle and establishes the form of the drifts to windward and leeward of the obstacle. The same procedure is applied in the analysis of silt deposits in rivers.

2. General mechanics

The earliest of Joukovsky's most significant works in the field of general mechanics was *Stability of Motion* 1882, the theme of his doctor's thesis.

Earlier investigations in the field had been concerned mainly with particular cases. The first attempts to formulate a general theory were made by Thompson and Tait, and Routh. Joukovsky's research differed both in point of view and method of analysis from Routh's *Treatise on the Stability of a Given State of Motion*, which appeared at almost the same time. Joukovsky's formulation of the question of stability of a conservative system retains its significance to the present day, even after the appearance of the famous work of Liapounoff on stability of motion.

The opening chapter of Joukovsky's work considers the motion of a point over a surface under the action of a force having a potential,

The system of curvilinear orthogonal coordinates on the surface is taken so that the trajectory of the point coincides with one of the coordinate lines. Like Thompson and Tait, Joukovsky sets up the equation of the trajectory of the disturbed motion on the basis of the principle of minimal action, the principle being presented in the Jacobi form. Retaining only the terms of the first order of smallness and introducing new variables, the author reduces the differential equation of the trajectory of disturbed motion to a linear equation of the second order. This equation may be understood as the equation of the motion of a representative point along a straight line under the action of a force proportional to the distance of the point from the origin of the coordinates and under the action of some other constant force. Joukovsky shows that the function which is the coefficient of proportionality in the expression for the force acting on the representative point is independent of the choice of orthogonal coordinates. He introduces a quantity related to this coefficient, and calls this quantity the measure of stability. The measure is also dependent upon the curvature of the surface. The introduction of this quantity is justified to a certain extent by the fact that the value of the quantity must be greater than zero for the motion of a point to be stable in the case of a very large group of types of motion. The motion will be the more stable the greater the function. Joukovsky calls that motion stable in which the distance between the trajectory of the disturbed motion and the trajectory of the given motion remains small.

This investigation was later carried further in Joukovsky's *Conditions of Finity of Integrals of Equation* $d^2y/dx^2+py=0$[56, I] (1892) and expanded by Liapounoff in *Problème général de la stabilité du mouvement*[1]—analyses of the case not considered in the first work, when the coefficient of proportionality in the expression for the force moving the representative point is a periodic function.

The first chapter ends with a number of examples of stability of motion of a point in a plane, including motion under the action of a central force, motion caused by the attraction of two centres according to Newton's law; as well as motion of a heavy point on the surface of a cone with an inclined axis, motion over a surface of revolution with a vertical axis, and motion by inertia over an arbitrary surface. Finally the author shows that the measure of stability is equal to the spherical excess of an infinitesimal triangle formed by three trajectories, divided by the area of the triangle.

Chapter two deals with the stability of motion of a point with two degrees of freedom. The problem is directly reduced to the problem of the first chapter, the author giving examples.

Chapter three is an investigation of the stability of motion of a system. In it Joukovsky relates the system to an orthogonal system of coordinates in a multidimensional space in such a way that the trajectory coincides with one of the coordinate lines. By means of a number of transformations the problem is reduced

[1] А. М. Ляпунов. Общая задача об устойчивости движения. Харьков. 1892. A. M. Liapounov. Problème général... Annales de Toulouse. S. 2. J. IX. 1907.

to a system of linear differential equations of the second order, of the same type as in chapter one. Here too the motion of the system will be stable if the coefficients of proportionality, like those in chapter one, are positive; and unstable if some of these coefficients are negative. The general equations are applied to the motion of a free material point, as illustrated by two examples. The first is an investigation of stability of motion of a point along a circle under the attraction of forces proportional to the n'th power of the distance to two centres on a line drawn through the centre of the circle at right angles to its plane. The second example treats of stability of motion of a point along an ellipse, caused by Newtonian attraction to two centres.

The final chapter deals with the very important problem of stability of steady motions, whose equations of disturbed motion have constant coefficients. The problem is reduced to a system of linear differential equations of the second order with constant coefficients, and is further reduced to the investigation of the roots of a determinant. If all of the roots are real and negative the motion will be stable. This property of the roots depends upon whether a certain quadratic form is always positive. This result was earlier obtained by Routh in another way.

As he does throughout his work, Joukovsky takes one of the coordinates as an independent variable, so that time is a function of this coordinate. The investigation of the change of time for the motion along the trajectory in the theory of stable motion leads to the same conclusions as formerly: any conservative disturbance produces an infinitesimal change of time in a stable steady motion; and an infinite change in case of non-conservative disturbance. As an example, Joukovsky examines the stability of motion of a heavy top spinning on a plane and the stability of motion of three material points acting on each other with a force proportional to any power of the distance between them. The latter problem is considered in the two cases when the points remain at the vertices of an equilateral triangle, and when they are located on a straight line. Joukovsky's analysis of the first case is somewhat simpler than that of Routh, which appeared earlier. The second case was investigated by Liouville for the Newton law. Joukovsky takes up the case of any power in the expression for the force. He finds that motion is always unstable in case of a negative power in the expression for the force; in case of a positive power, the motion may be stable.

Joukovsky's *Geometrical Interpretation of the Case of Motion of a Solid About a Fixed Point Treated by Sofia Kovalevskaya* [67,II] (1896), is among the most profound investigations in analytical mechanics. The interpretation is based on the geometrical significance of the two hyper-elliptic integrals in terms of which Kovalevskaya expresses all the quantities determining the position of the solid. Joukovsky introduces a system of curvilinear isothermic coordinates connected with two hyper-elliptic integrals, and shows that these variables may be expressed in hyper-elliptic functions of time. This enables him to arrive at the Kovalevskaya theorems. In his analysis Joukovsky employs the properties of motion of the end of the projection of the angular velocity in the plane of equal radii of gyration, the motion being determined in the system of coordinates spoken of above. Employing this motion, the motion may be found in the solid of a point on a vertical line at unit distance from the fixed point. This yields the locus of the verticals in the solid, which Joukovsky calls the cone of the verticals.

He then proceeds to the next geometrical interpretation of the motion. In the Kovalevskaya motion of the solid, the cone of the vertical slides over the vertical passing through the fixed point in accordance with a law for the motion of the end of the projection of equal radii of gyration, while the plane defined by the vertical and a point on the axis of the ellipsoid of gyration corresponding to an unequal moment of inertia at unit distance from the fixed point revolves about this vertical with a definite angular velocity. The expression for this velocity is given by the author.

Joukovsky is the author of an elegant geometrical investigation *The Hess Loxodromic Pendulum*[59,I] (1893). This is a geometrical interpretation of the Hess case of rotation of a solid about a fixed point, obtained when the centre of gravity of the solid lies on the perpendicular drawn from the point of support to the plane of the circular cross-section of the gyration ellipsoid, and the principal moment of momentum at the initial moment lies in the plane of that circular section. Joukovsky gives a description of a pendulum which he constructed, to illustrate the motion of a solid in the Hess case.

Joukovsky gives a geometrical explanation of the discrepancy in the methods employed by Jacobi and Hamilton, in his *Characteristic Functions of Jacobi and Hamilton* [16,I] (1883).

In his proof of Hamilton's theorem, which makes it possible to find the integrals of the differential equations of motion by means of the characteristic function, Jacobi demonstrated that the function need satisfy only one of the equations, and that there is no need of a second equation, as in Hamilton's proof. Joukovsky shows that Hamilton's method of proof may be employed to deduce Jacobi's form of his theorem.

An interesting work in the field is his *Determination of the Potential Function for a Given Family of Trajectories*[47,I] (1890). By means of the centripetal force formula for motion along a surface and in space, Joukovsky determines the potential function for which a material point acted on by forces characterised by the function will describe the given family of trajectories on the given surface. His results, in the form of a theorem, are included in such classic courses in mechanics as E. T. Whittaker's well known *Treatise on the Analytical Dynamics of Particles and Rigid Bodies*.

The work *Relation Between the Problem of Motion of a Material Point and That of Equilibrium of a Flexible Cord* [8,I] (1879) is one of those studies which at the very outset of his scientific activity characterise the work of Joukovsky, who often found analogies between widely differing phenomena in mechanics. Here the differential equations of motion of a material point are transferred into the equations of equilibrium of a cord acted upon by forces having a force function, by means of special substitutions.

In 1911, Joukovsky published an important work *Reduction of the Dynamic Problem of a Kinematic Chain to the Problem of a Lever*[125,I]. In it he gives a method of solving various dynamic problems of the kinematic chain with one degree of freedom, by reducing them to problems of a lever in the form of a rigid frame having the shape of Mohr's diagram of velocities and supported at the pole.

The paper opens with instructions for drawing up the velocity diagram and the diagram of centripetal forces to determine the velocities and accelerations of points in a kinematic chain. Examples are given of the construction of diagrams and the solution of a number of velocity and acceleration problems.

The auxiliary lever of the kinematic chain is then introduced, the term being applied to a statically determinate frame having the shape of the diagram of the velocities of the chain obtained when the angular velocity of a "main link" is equal to unity. The dynamic relationship between the auxiliary levers and the kinematic chain is expressed by the following basic theorem:

The condition of equilibrium of a kinematic chain subjected to forces may be replaced by corresponding conditions of an auxiliary lever subjected to equal and parallel forces applied at corresponding points. Simultaneously, the elastic forces in the chain links will be equal to those in the corresponding elements of the auxiliary lever.

The general theory is illustrated by a number of examples, among them the calculation of the period of small oscillations of a kinematic chain and the determination of the length of a mathematical pendulum equivalent to this chain; determination of the elastic forces in a moving kinematic chain, in the presence of forces of friction; and calculation of frictional loss in the hinge of a chain.

Joukovsky made significant contributions to methods of solution of problems in applied mechanics. Illustrative of such applied investigations is *Conditions of Equilibrium of a Solid Lying Flat on a Fixed Plane and Free to Move over this Plane with Friction*[76,1] (1897). In it, Joukovsky proves a number of theorems relating to friction forces developed by rotation of a solid about a centre.

Joukovsky defines the point where the friction forces are reduced to a torque as the pole of friction; and the locus of those centres of rotation, for which the resultant moment of all the forces of friction is constant, as the line of equal moments.

The first theorem states that the resultant of the friction forces is parallel to the tangent at the centre of rotation to that line of equal moments which passes through the centre of rotation; and is equal to the derivative of the moment of friction along the normal to the line of equal moments.

Theorem two states that there exists only one pole of friction. The moment of force of friction is a minimum for this point, which is within the area of contact; for any other centre of rotation the friction forces have a resultant.

Theorem three states that the moment of forces of friction with respect to any point for rotations about different centres will be a maximum when the centres of rotation and the given point coincide.

The theorems yield the following conditions of equilibrium:

The necessary and sufficient condition for the equilibrium of a solid lying flat on a fixed plane is that the force acting on the solid along a line drawn in the fixed plane should not exceed the force of friction of contact area along the same line. If there is a torque acting on the solid in the fixed plane, the necessary and sufficient condition of equilibrium is that the torque should not exceed that of the couple of friction forces resulting from the rotation of the contact area about the pole of friction. The results obtained by Joukovsky were the starting point of the development of methods of calculating friction gears.

In a critical paper *Slipping of a Belt on a Drive Wheel* (1894), Joukovsky upholds N. P. Petrov's view on the existence of ranges of the contact arcs along which slipping does not take place, and describes an apparatus which he constructed to demonstrate the existence of non-slip ranges.

In a short paper on *Flat Sifting*[68,VIII] (1896), Joukovsky suggests a system of suspension of the sieve consisting of three double cones rolling on conical supports. He formulates a theory for the motion of the particles in the sieve under the action of centrifugal forces and friction. He finds the trajectory of motion of the particles and describes the rôle of the teeth at the bottom of the sieve.

In a paper entitled *Flexible Shaft of a Laval Turbine and Shafts Mounted in Oscillating Bearings* [85,I] (1899), Joukovsky extends the theory of the turbine shafts as developed by Föppl and Stewart to the case when the shaft is mounted in oscillating bearings. It is found that mounting the flexible shaft in oscillating bearings yields approximately the same results as in ordinary mountings, the stability of motion with oscillating bearings being the greater, the more the angular velocity exceeds the critical angular velocity.

Joukovsky's *Strength of a Bicycle Wheel* [94,VIII] (1902) is a study of a wheel with a large number of thin spokes, radial pressure alone being applied to the rim of the wheel, and extension or compression forces to the spokes. The problem is simplified by the assumption that pressure of the spokes is uniformly distributed over the rim, and in each element of length proportionally to the change in radius of the rim axis. Applying the equilibrium conditions for a thin curved beam, the author reduces the problem to two simultaneous differential equations of the second and fourth order for the increments of radius and the increments in the centre angle of the points of the axis of the rim. The equation of the fourth order is the ordinary linear differential equation with constant coefficients, and is integrated directly, yielding the expressions for the increments in the radius. This makes integrating the second equation containing the increments of the angle a simple matter. In interpreting the results, Joukovsky simplifies the formulae somewhat, by neglecting some of the smaller magnitudes entering into the expressions. Joukovsky concludes that the length of the spokes undergoes periodic change, the changes damping rapidly from the shortest point at contact with the earth and rising along the circumference of the wheel.

The great variety of Joukovsky's works in the field of technical mechanics includes, for example, studies in *Pressure Exerted by a Piston on the Cylinder of Gnome Motors* [120,VIII] (1911), *Vibrations of a Locomotive on Shock-Absorbing Springs* [169,VIII] (1937), *Dynamics of the Automobile* [156,VIII] (1923), *Action of Couplings in Motion of a Train from Rest* [151,VIII] (1919). Limitation of space does not permit a discussion here of these interesting studies.

Limitation of space as well does not permit a detailed description of Joukovsky's work in astronomy, among which we find such interesting studies as *Simplified Exposition of Gauss' Method of Determining Planet Orbits* [18,I] (1883), giving a concise exposition of Gauss' method based on geometrical considerations appreciably simplifying the theory; *Graphical Solution of the Fundamental Equation in the Calculation of Planetary Orbits* [20,IX] (1883), giving a graphical solution of an equation of the eighth order in the theory of determination of orbits, for the case when the time interval between observations is small. Another interesting work in the field is *A Proof of Lambert's Theorem* [25,IX] (1884). Joukovsky's proof is based on the variation of action for an infinitesimal variation of orbit. *In Construction of Syndynamical and Synchronic Curves* [24,IV] (1884) Joukovsky deduces formulae for the construction of curves in the theory of comet tails, which Bredikhin who introduced them called syndynamical and synchronic curves. Joukovsky gives formulae for the construction based both on the approximate and the exact methods. In *Solution of a Problem in the Comet Theory*[23,IX](1884) Joukovsky determines the change in the geocentric position of particles in the comet tail expelled from the nucleus at a given previous time by a repulsive force, for an infinitesimal change in the force.

3. Aerodynamics

The work of Joukovsky in aerodynamics reflects the entire development of the science of aviation; in many cases, his work formed the basis for important steps forward in the field. The present review therefore preserves chronological order throughout, including works which may for some be of purely historical interest.

The first of Joukovsky's studies in the field, entitled *On the Flight Theory*[39,VI] appeared as far back as 1890. It is a discussion of the origin of the propulsive force acting on a body immersed in a fluid, considering that the body can develop only equal and opposite internal forces. Joukovsky proves that in the absence of friction the propulsive force in a perfect fluid can arise only owing to a change in the motion of the fluid, and that the total work throughout a period of the change is equal to zero. That absence of circulation must be introduced into the above formulation was later established by Joukovsky himself. Joukovsky goes on to show that the formation of both free stream lines and friction may account for the appearance of lift force. The conclusion that Joukovsky draws from this first attempt is an interesting step in the development of scientific thought. For here Joukovsky agrees with Brillouin that there can hardly be any basis for assuming the existence of free stream lines stretching out behind the body to infinity, and that their forming closed contours behind the body may produce only a zero pressure. Joukovsky holds the view that the origin of the propulsive force must be sought in friction between the body and the fluid.

Later Joukovsky himself proved that one of the chief sources of the lift force lies in the bound vortices which exist as well in perfect fluids; that the significance of friction is not so large that it cannot in great part be neglected.

In his work *Soaring of Birds*[46,VI](1891) Joukovsky carries out a theoretical study of gliding, that is, the type of flight in which the bird's wings do not flap. He shows that there may be two types of soaring, in which the bird loses altitude or slides through the air, and when the bird maintains its altitude or gains in altitude. In the preliminary discussion of existing theories, Joukovsky points out a number of points lacking in clarity and contradictory. In the main section of the work he discusses soaring of birds in still air and the influence on such flight of motion of air flowing in horizontal layers at differing speeds, gusts of wind and wind blowing with a slightly upward motion . Joukovsky treats the bird as a thin plate, and sets up differential equations of the motion of the bird's centre of gravity when its path is a flat curve and when it is a spatial curve. The study of the curves explains a number of phenomena noted by observers of the flight of birds.

In a short paper *The Most Efficient Incidence of a Wing*[74,X](1897) Joukovsky shows how the observations of Otto Lilienthal in air resistance may be employed in solving the problem. A simple geometrical study based on the Lilienthal curve proves that the most efficient incidence, i. e., the angle for which the work done in horizontal translation will be a minimum is approximately 15 degrees. However, in the work Joukovsky does not discuss the influence of the other parts of the plane and the deformation of the wings.

The next step in the development of Joukovsky's conceptions of the forcse acting on a body in flight may be seen in his classic work *On Bound Vortices*, of which he spoke at the Moscow Mathematical Society in 1905, and which was published in the following year. Published in the same year was the related work *Falling Through the Air of Light Elongated Bodies Rotating about their Longitudinal Axes*. Here Joukovsky speaks for the first time of the possibility of circulation arising due to the rotation of the body. It was not until four years later, in 1910, that the true cause of circulation was found. This part of the problem was carried out in collaboration with Joukovsky's pupil, Chaplygin, who succeeded in finding the wing shape round which circulation will arise; and the two scientists could then proceed to the calculation of the lift force and moment depending on the incidence. This then was the long and difficult path leading to the formulation of the famous Joukovsky—Chaplygin postulate, which may be said to have marked the inception of aerodynamics as an independent science.

A most important work[1] in the field of aerodynamics is *Geometrical Investigation of the Kutta Flow*[128, V](1910, 1911). At the beginning of the work Joukovsky speaks of the investigation of Kutta (1902), who found the stream function for the non-vortex two-dimensional flow of an incompressible fluid about an arc of a circle and having a velocity at infinity directed along the chord of the arc. Joukovsky goes on to Chaplygin's method (1902) of determining a non-vortex two-dimensional flow about certain contours, among them the Kutta case. Joukovsky speaks of the explanation of the lift force as arising due to vortex filaments suggested by Lanchester (1907), who, however, did not give the magnitude or the direction of the pressure. Joukovsky then quotes his theorem (1906) for the pressure exerted by a non-vortex flow having a velocity at infinity, on a contour around which there is a circulation. The direction of this pressure is obtained by rotating the velocity vector through ninety degrees in a direction counter to the circulation. Employing this theorem he proves both the Kutta and the Chaplygin formulae and extends the results to a rectilinear plate. He discusses contours whose shape approaches the curved and rectilinear wing with a widening at the front end.

Joukovsky then goes on to prove the very important theorem that in conformal transformations of a stream flowing about a contour there is no change in the magnitude of the circulation about the contour enclosing the original shape.

This is followed by a simple and elegant geometrical exposition, based on Chaplygin's formula, of how the entire area of a plane external to a circular arc-shaped aperture may be conformally represented on the entire area of another plane external to a circular aperture in the plane. He employs another conformal representation for a plane with a rectangular finite aperture on another plane with a circular aperture. Joukovsky applies his method to the construction of various profiles, and to the profile now well known as the Joukovsky wing.

[1] Extracts from this work were published as Geometrische Untersuchugen über die Kutta'sche Strömung by Joukovsky in Zeitschrift für Flugtechnik und Motorluftschiffahrt in 1910 (Heft 22) and in 1912 (Heft 6).

Joukovsky returns to these questions in two works, *Calculation of the Pressure of a Two-Dimensional Flow on a Contour of which a Segment of a Straight Line is the Limit*[123,V](1911); and *Lifting Planes of the Antoinette Type*[130,V](1912). He bases his investigation on Chaplygin's convenient formula for the resultant moment of pressure on a contour and applies the results to determining the lift force and centre of pressure of the Joukovsky wing and a number of other aerofoils, among them the profile which Joukovsky calls the Antoinette type and which was later discussed by Karman and Trefftz in 1917.

We are indebted to Joukovsky for great achievements in another basic problem of the theory of flight—the propulsive force of the propeller. The problem is not new in hydrodynamics, related as it is to the propulsive force developed by ships' screws. Joukovsky at various times studied aspects of the question in early works, *On Winged Propellers*[80,VI](1898), *On the Efficiency of the Helicopter*[100,VI](1904) and *Theory of a Screw with Numerous Blades*[108,VI](1907). However, his greatest contribution in the field is his hydrodynamic theory of the propeller. In four papers, entitled *Vortex Theory of the Propeller*,[129,VI] (1912, 1914, 1915, 1918), he evolves an entirely new theory of propellers, containing the theoretical bases for the study of all problems connected with the designing of propellers. It is of interest to note that the Joukovsky theory of the propeller, as well as Prandtl's theory of the finite wing were made possible by Joukovsky's investigations of lift force, both theories being the natural outcome of the famous Joukovsky theorem.

However, the step from the theory of the wing to the theory of the propeller was no easy task. Joukovsky's theoretical results are based on numerous preliminary experiments. His work in the field enabled him to determine the efficient shape of the propeller, compute all its elements and to construct the propeller, which came to be called the NEJ propeller (the initials of Joukovsky's name). Experiment proved the correctness of the theoretical calculations and NEJ propellers have been used in practice. The basic results of Joukovsky's vortex theory of the propeller may be found today in any textbook on the subject.

Joukovsky's interests, particularly in later years, embraced almost every branch of engineering connected with aviation. In *Investigation of the Stability of Structural Parts of Airplanes*[148,VIII](1918), he discusses the stability of thin bars. Joukovsky was one of the first to draw attention to the importance of wind tunnel problems in *Entrance Cones and Diffusers of Wind Tunnels,* in which he speaks of the importance of determining the shape of diffusers which will not distort the streams. Joukovsky was, moreover, the first to speak of the aerodynamic efficiency of the wind tunnel. In *Bombing Practice*[141,V] (1916) he gives elements of the theory of sighting apparatus. He studied questions of gas dynamics, among them *An Analogy between the Motion of Heavy Liquid in a Narrow Channel* and the *High Velocity Motion of Gas in a Pipe*[160,VIII] (1925), *Motion of an Air Wave with Supersonic Velocity*[153,VI] (1920).

Throughout the remarkable variety of his researches, Joukovsky was the true pioneer and investigator. In the complex phenomena of nature he sought and found the action of basic laws, calculated their effects and indicated the path leading to their application in practice.